# A note on Borromean correlations in multipartite quantum systems

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**Abstract**

If a pure state of a multipartite quantum system is Borromean, that is, such that its density matrix becomes product after tracing out any its component then the initial state is product itself. This shows the essentially classical nature of Borromean correlations which can not be achieved by entangled pure states.

A random vector $(A_1, \dots, A_n)$ of length $N$ is said to be BORROMEAN[1] CORRELATED if the distribution obtained by "forgetting" (or, speaking quantum, tracing out) any one of its components is a product one. As an example of such vector when all its components $A_i$ are two-valued consider a classical register $(A_0, \dots, A_N)$ of $N+1$ bits. Let each bit $A_i$ $(i = 1, \dots, N)$ takes independently its values 0 and 1 with probabilities $p_i$ and $q_i$, respectively, while the 0-th bit contains the checksum of all the bits:

$$\begin{cases} A_1, \dots, A_N & \text{are independent} \\ A_0 = A_1 \oplus \cdots \oplus A_N \end{cases} \tag{1}$$

Then

- the set $(A_0, \dots, A_N)$ of random variables is not independent
- any marginal distribution $(A_0, \dots, \widehat{A_i}, \dots, A_N)$ which appears when any bit $i = 0, \dots, N$ of the register is traced out is a distribution of $N$ independent variables.

[1]The symbol of three rings linked in such a way that taking out any one makes the rest disjoint was used in the Renaissance as an heraldic device for the Borromeas family of Italy.

In classical probability, for a random vector to be Borromean correlated, it should necessarily be in a mixed state. It is known that in quantum mechanics pure states still can give rise to random correlations (which are seen from outside as sharing a secret variable [2]), and one could expect that there exists such a pure quantum state, for which we can write an analog of Borromean correlations. It is shown in this note that no pure state of a multipartite quantum system can possess Borromean correlated density matrix.

I consider a composite quantum system $\mathfrak{S}$ consisting of $N$ components $\mathfrak{s}_r$ each described by its state space $\mathcal{H}_r, \quad r = 1, \dots, N$ each of which has dimension $d$. The overall state space $\mathcal{H}$ of $\mathfrak{S}$ is the tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$$

Let $\rho$ be a density matrix of a state of of $\mathfrak{S}$. I call it BORROMEAN if for any $r = 1, \dots, N$ its partially traced (with respect to the subsystem $\mathfrak{s}_r$) density matrix is product:

$$\breve{\rho} = \mathrm{Tr}_r\, \rho = \rho_1 \otimes \cdots \otimes \rho_{r-1} \otimes \rho_{r+1} \otimes \cdots \otimes \rho_N \tag{2}$$

The message of this note is contained in the following

**Statement.** *Let $|\Psi\rangle$ be a pure state of the composite system $\mathfrak{S}$ such that its density matrix $\rho = |\Psi\rangle\langle\Psi|$ is Borromean (2). Then the state $|\Psi\rangle$ is product.*

*Proof* is essentially based on the generalised Scmidt decomposition introduced in [1]. Given a state $|\Psi\rangle$, one can always choose such a product basis $\{|\psi_i^{(r)}\rangle \mid \quad r = 1, \dots, N;\ i = 1, \dots, d\}$ in $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ that in the decomposition

$$|\Psi\rangle = \sum_{i_1 \cdots i_N} c_{i_1 \cdots i_N} |\psi_{i_1}^{(1)}\rangle \cdots |\psi_{i_N}^{(N)}\rangle \tag{3}$$

the coefficients $c_{i_1 \cdots i_N}$ have the properties:

$$c_{jii\cdots i} = c_{iji\cdots i} = \cdots = c_{ii\cdots ij} = 0 \quad \text{if} \quad 1 \le i < j \le d \tag{4}$$

$$|c_{ii\cdots i}| \ge |c_{j_1 \cdots j_n}| \quad \text{if} \quad i \le j_r\,;\ r = 1, \dots, N \tag{5}$$

Write down the density matrix

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{i_1 \cdots i_N} \sum_{j_1 \cdots j_N} c_{i_1 \cdots i_N} \overline{c}_{i_1 \cdots i_N} |\psi_{i_1}^{(1)}\rangle \cdots |\psi_{i_N}^{(N)}\rangle\langle\psi_{i_1}^{(1)}| \cdots \langle\psi_{i_N}^{(N)}|$$

and take its partial trace $\breve{\rho} = \mathrm{Tr}_r\, \rho$ with respect to the subsystem $\mathfrak{s}_r$. Then the matrix coefficients $\breve{\rho}_{i_1\cdots i_{r-1}i_{r+1}\cdots i_N,\, j_1\cdots j_{r-1}j_{r+1}\cdots j_N}$ read:

$$\breve{\rho}_{i_1\cdots i_N,\, j_1\cdots j_N} = \sum_{i_r} c_{i_1\cdots i_{r-1}i_r i_{r+1}\cdots i_N}\overline{c}_{i_1\cdots j_{r-1}i_r j_{r+1}\cdots i_N} \tag{6}$$

Since $\rho$ is assumed to be Borromean, the density matrix $\breve{\rho}$ is product, that is, there is a set $\breve{\rho}^{(1)}, \ldots, \breve{\rho}^{(r-1)}, \breve{\rho}^{(r+1)}, \ldots, \breve{\rho}^{(N)}$ of $d \times d$ density matrices in each $\mathcal{H}_i$ that

$$\breve{\rho}_{i_1\cdots i_N,\, j_1\cdots j_N} = \breve{\rho}^{(1)}_{i_1 j_1} \cdot \ldots \cdot \breve{\rho}^{(N)}_{i_N j_N} \tag{7}$$

If we set all $i_1, \ldots, i_N,\ j_1, \ldots, j_N = 1$ then it follows from (4) that all the summands in (6) but one vanish, namely $\breve{\rho}_{11\cdots 11,\, 11\cdots 11} = c_{11\cdots 11} \cdot \overline{c}_{11\cdots 11} > 0$, therefore from (7) we conclude that

$$\breve{\rho}^{(1)}_{11} > 0, \ldots, \breve{\rho}^{(N)}_{11} > 0 \tag{8}$$

For any $s \neq r$, $i_s > 1$ it follows from (4) that all summands in (6) are 0, therefore we infer from (7) and (8) that

$$\forall s \neq r,\ \forall j_s > 1 \quad \breve{\rho}^{(s)}_{i_s 1} = \breve{\rho}^{(s)}_{1 i_s} = 0 \tag{9}$$

Furthermore, for any multi-index $i_1 \cdots i_N$ ($i_r$ excluded) we have

$$\breve{\rho}_{i_1\cdots i_N,\, 11\cdots 11} = \breve{\rho}^{(1)}_{i_1 1} \cdot \ldots \cdot \breve{\rho}^{(N)}_{i_N 1} = 0$$

if at least one $i_s$ differs from 0. Now consider the right-hand side of the expansion (6) for this case. Again it follows from (4) that there is at most one non-zero summand in (6), that is:

$$\breve{\rho}_{i_1\cdots i_N,\, 11\cdots 11} = c_{i_1\cdots i_N} \cdot \overline{c}_{11\cdots 11}$$

Since we can repeat our reasoning for any $r$, we have proved that for any multi-index $i_1 \cdots i_N$ ($r$ now included) the coefficient $c_{i_1\cdots i_N}$ in (3) vanishes whenever at least one $i_r = 1$ and at least one $i_s > 1$.

If $c_{22\cdots 22} = 0$ then it follows from (5) that all the coefficients in (3) are 0 and we are done: $|\Psi\rangle = |\psi^{(1)}_1\rangle \cdots |\psi^{(N)}_1\rangle$ is a product state. So, suppose $c_{22\cdots 22} \neq 0$. In this

case, applying (4) for $i = 2$ and repeating all the above reasoning we get that for any multi-index $i_1 \cdots i_N$ the coefficient $c_{i_1 \cdots i_N}$ in (3) vanishes whenever at least one $i_r = 2$ and at least one $i_s > 2$. Repeating the procedure up to $d$ (the dimension of each $\mathcal{H}_s$) we see that the only nonzero coefficients in the expansion (3) are of the form $c_{ii\cdots ii}$, $i = 1, \ldots, d$. Now return to the formula (6) for the reduced density matrix. We see that its only non-vanishing elements are of the form $\breve{\rho}_{ii\cdots ii,ii\cdots ii}$ which means that it is diagonal. Since $\breve{\rho}$ is a pure state, $c_{11\cdots 11} = 1$ and all others $c_{ii\cdots ii} = 0$, therefore $|\Psi\rangle = |\psi_1^{(1)}\rangle \cdots |\psi_1^{(N)}\rangle$ is always a product state.

So, tracing out our pure state $\rho$ we get a pure *product* state. This, in turn, can happen only when the pure state $\rho$ is product itself.

**Concluding remarks.** It was demonstrated that there are correlations in quantum systems which are of purely classical nature, that is, they can not be provided by any pure quantum state.

Note that the condition (2) for density matrices to be Borromean is stronger than its classical analogue. If we weaken the condition (2) and require the Borromean correlations not for the density matrix but only for a given fixed set of local observables, then it can be achieved by a pure quantum state. This can be shown by an example. Consider the following pure state of $N + 1$ qubits:

$$|\Psi\rangle = \frac{1}{\sqrt{2^N}} \sum_{i_1,\ldots,i_N=0,1} |i_1 \oplus, \ldots, \oplus i_N\rangle |i_1\rangle \cdots |i_N\rangle$$

then the expectation values of the collection of observables $|1\rangle\langle 1|$ in each qubit has exactly the same distribution as shown the example (1).

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